Economic Growth, June 2010.

Christian Groth

A suggested solution to the problem set at the exam in Economic Growth, June 3, 2010

(3-hours closed book exam)[1]

As formulated in the course description, a score of 12 is given if the student's performance demonstrates precise understanding of the concepts and methods needed for analyzing the factors that matter for economic growth.

1. Solution to Problem 1 (25 %)

Each country has a Cobb-Douglas production function

$$Y_t = K_t^{\alpha} H_t^{\beta} (T_t L_t)^{1-\alpha-\beta}, \qquad 0 < \alpha < 1,\ 0 < \beta < 1. \tag{*}$$

The gross investment rates in the two types of capital are a fraction s_K and s_H of GDP, respectively, so that (for a closed economy) we can write

$$\dot{K} = s_K Y - \delta K, \tag{1.1}$$

$$\dot{H} = s_H Y - \delta H, \tag{1.2}$$

assuming the depreciation rate is the same for both types of capital. The technology $T_t = T_0 e^{gt}$, $g \geq 0$, is the same for all countries in the sample (apart from a noise term affecting T_0). And within a country all workers have the same amount of human capital, equal to $h_t \equiv H_t/L_t$. Mankiw, Romer, and Weil (from now MRW) find that $\alpha = \beta = 1/3$ fits the data well.

a) We have

$$\begin{aligned} Y_t &= K_t^{\alpha}(h_t L_t)^{\beta}(T_t L_t)^{1-\alpha-\beta} = K_t^{\alpha} {h_t}^{\beta} T_t^{1-\alpha-\beta} L_t^{1-\alpha} \\ &= K_t^{\alpha}\left(T_t^{\frac{1-\alpha-\beta}{1-\alpha}} h_t^{\frac{\beta}{1-\alpha}} L_t\right)^{1-\alpha} = F(K_t, X_t L_t), \end{aligned}$$

[1]The solution below contains *more* details and more precision than can be expected at a four hours exam. The percentage weights should only be regarded as indicative. The final grade will ultimately be based on an assessment of the quality of the answers to the exam questions in their totality.

where $X_t = T_t^{\frac{1-\alpha-\beta}{1-\alpha}} h_t^{\frac{\beta}{1-\alpha}}$.

b) In (*) human capital and man hours are written as if they could be separated such that the firm hires *three* production factors, K, H, and L. But human capital is embodied in the workers and when hiring a worker also his or her human capital is hired. Under perfect competition the firm will hire hours up to the point where

$$\frac{\partial Y_t}{\partial L_t} = F_2 X_t = (1-\alpha) K_t^{\alpha} \left(T_t^{\frac{1-\alpha-\beta}{1-\alpha}} h_t^{\frac{\beta}{1-\alpha}} L_t \right)^{-\alpha} T_t^{\frac{1-\alpha-\beta}{1-\alpha}} h_t^{\frac{\beta}{1-\alpha}} = w_t,$$

where w_t is the hourly real wage for workers with human capital h_t.

c) We have

$$\tilde{y} \equiv \frac{Y}{TL} = \frac{K^{\alpha}(hL)^{\beta}(TL)^{1-\alpha-\beta}}{TL} = \left(\frac{K}{TL}\right)^{\alpha} \left(\frac{h}{T}\right)^{\beta} \equiv \tilde{k}^{\alpha}\tilde{h}^{\beta} \to (\tilde{k}^*)^{\alpha}(\tilde{h}^*)^{\beta},$$

for $t \to \infty$. Hence, in the long run

$$y_t \equiv \frac{Y_t}{L_t} \equiv \tilde{y} T_t = (\tilde{k}^*)^{\alpha}(\tilde{h}^*)^{\beta} T_t = (\tilde{k}^*)^{\alpha}(\tilde{h}^*)^{\beta} T_0 e^{gt}.$$

We see that the long-run growth rate of y is g, the rate of technical progress.

In the absence of technical progress, $\dot{y}/y = 0$ in the long run. So human capital accumulation (alone) does not drive growth in the long run. (But it can be said that human capital accumulation "contributes" to maintaining the per capita growth rate g in the sense that if h were only kept constant, y would in the long run grow only at the rate $[(1-\alpha-\beta)/(1-\alpha)]\, g < g$.)

d) We now replace (*) by

$$Y_t = K_t^{\alpha}(h_t L_t)^{1-\alpha} \qquad (**)$$

and assume that $T_t \equiv 1$ in (*). Whereas (*) has decreasing returns to producible inputs ($\alpha + \beta < 1$), (**) has constant returns to producible inputs. Hence accumulation of H, together with K, can generate persistent per capita growth in spite of $g = 0$. So (**) gives a different answer to the last question in c).

e) One problem with the Barro & Sala-i-Martin set-up is that there is no theoretical reason to believe that the exponent on h should be exactly $1 - \alpha$ when human capital formation obeys (1.2). Here the replication argument is of no help. And also from an empirical point of view is it difficult to find support for (**). The MRW finding $\alpha = \beta = 1/3$ exemplifies this.

2. Solution to Problem 2 (65 %)

Firm i ($i = 1, 2, ..., M$) in the competitive manufacturing sector has the production function

$$Y_i = A\left(\sum_{j=1}^{N} x_{ij}{}^{\alpha}\right) L_i^{1-\alpha}, \qquad A > 0,\ 0 < \alpha < 1. \tag{2.1}$$

The labor force is $L = \sum_i L_i$ and is constant.

a) Firm i chooses x_{ij} such that the marginal productivity of intermediate j equals its price P which is the same for all $j = 1, ..., N$. And since the intermediates enter the production function in a completely symmetric way, the implied N first-order conditions for profit maximization are satisfied only if the quantity x_{ij} is the same for all j.

Substituting $x_{ij} = x_i$ for all j into (2.1) gives

$$Y_i = ANx_i{}^{\alpha}L_i^{1-\alpha}. \tag{2.2}$$

b) The result (2.2) can be written

$$Y_i = A(Nx_i)^{\alpha}(NL_i)^{1-\alpha} \equiv f(Nx_i, N, L_i),$$

where Nx_i is the total input of intermediate goods. We see that

$$\frac{\partial Y_i}{\partial N}|_{Nx_i=\text{const.}} = f_2(Nx_i, N, L_i) > 0.$$

This says that for a given L_i and a given total input, Nx_i, of intermediate goods, the higher the number of varieties (with which follows a lower x_i of each intermediate), the more productive is this total input. "Variety is productive". There are "gains to division of labor and specialization in society". Thus the number of input varieties, N, can be interpreted as a measure of the level of technical knowledge.

c) Profit maximization wrt. the labor input implies, by (2.2),

$$\frac{\partial Y_i}{\partial L_i} = (1-\alpha)ANx_i{}^{\alpha}L_i^{-\alpha} = (1-\alpha)AN\left(\frac{x_i}{L_i}\right)^{\alpha} = w, \tag{2.3}$$

where w is the real wage. All firms in the manufacturing sector will thus choose the same x_i/L_i. In view of (2.3) and (2.2) it follows that

$$y_i \equiv \frac{Y_i}{L_i} = AN\left(\frac{x_i}{L_i}\right)^{\alpha} = \frac{w}{1-\alpha}, \tag{2.4}$$

i.e., the same for all i. (That the firms have the same production function does not necessarily imply that x_i and L_i are the same across firms, only that x_i/L_i is the same.)

d) The reason that

$$\frac{x_i}{L_i} = \frac{X_m}{L} \tag{2.5}$$

is the following. When the intermediate j-to-labor ratio is the same for all firms i = 1,..., M, it is also the same as the economy-wide ratio, which is X_j/L, where X_j is the aggregate demand for intermediate j. In turn, because of symmetry, this aggregate demand is the same for all j and must equal the aggregate supply of each intermediate good (demand = supply). This aggregate supply is denoted X_m, where the subscript m indicates "monopoly supply".

Although such a verbal explanation is certainly sufficient, we may give a more formal derivation: Let the common value of x_i/L_i be denoted z. Then $x_i \equiv zL_i$, $i = 1, \ldots, M$, and aggregate demand for intermediate j satisfies

$$X_j = \sum_{i=1}^{M} x_{ij} = \sum_{i=1}^{M} x_i = \sum_{i=1}^{M} zL_i = zL.$$

Thus

$$\frac{x_i}{L_i} \equiv z = \frac{X_j}{L} = \frac{X_m}{L},$$

where X_m is the monopoly supply of intermediate j and the last equality reflects demand = supply.

e) Substituting (2.5) into (2.4) and summing gives

$$\begin{aligned} Y &= \sum_{i=1}^{M} y_i L_i = \sum_{i=1}^{M} AN(\frac{X_m}{L})^\alpha L_i = AN(\frac{X_m}{L})^\alpha \sum_{i=1}^{M} L_i = ANX_m^\alpha L^{1-\alpha} \\ &= A(NX_m)^\alpha (NL)^{1-\alpha}. \end{aligned}$$

The aggregate production function in manufacturing at time t can thus be written

$$Y_t = AX_t^\alpha (N_t L)^{1-\alpha}, \qquad A > 0, 0 < \alpha < 1, \tag{2.6}$$

where $X_t = N_t X_m$. (As the question is stated, it is not necessary to also show that $X_m = (\alpha^2 A)^{1/(1-\alpha)} L$, but this result follows from the first-order conditions mentioned in a) combined with (2.5) and $P = 1/\alpha$.)

f) The aggregate manufacturing output (2.6) is used partly for replacing the intermediate goods used up in the production of Y_t, partly for consumption, C_t, and partly for investment in R&D:

$$Y_t = X_t + C_t + R_t. \tag{2.7}$$

The invention production function is

$$\dot{N}_t \equiv \frac{dN_t}{dt} = \frac{R_t}{\eta}, \qquad \eta > 0, \ \eta \text{ constant.} \tag{2.8}$$

The "growth engine", as specified by (2.6), (2.7), and (2.8) taken together, is likely to be technologically capable of generating fully endogenous growth. The reason is that the manufacturing output (2.6), part of which constitutes the R&D investment, is produced under constant returns to scale wrt. producible inputs, X and N.

g) The implied Keynes-Ramsey rule for the individual household is

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho), \tag{2.9}$$

where r_t is the real interest rate at time t.

h) Along an equilibrium path with positive R&D, the value of an innovation, V_t, must equal the cost:

$$V_t = \eta. \tag{2.10}$$

The rate of return to financial wealth placed in an equity share of an innovative firm is $(\pi_t + \dot{V}_t)/V_t$, and on the loan market it is r_t. Thus, the no-arbitrage condition is

$$\frac{\pi_t + \dot{V}_t}{V_t} = r_t \text{ for all } t. \tag{2.11}$$

In view of (2.10), $\dot{V}_t = 0$. Substituting into (2.11) gives the equilibrium interest rate:

$$r_t = \frac{\pi_t}{\eta} = (\frac{1}{\alpha} - 1)\frac{X_m}{\eta} = (\frac{1}{\alpha} - 1)\frac{L}{\eta}(\alpha^2 A)^{\frac{1}{1-\alpha}} \equiv r, \tag{2.12}$$

where the third equality follows from $X_m = (\alpha^2 A)^{1/(1-\alpha)} L$. The interest rate is thus time independent. As a result also the per capita consumption growth rate,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left[(\frac{1}{\alpha} - 1)\frac{L}{\eta}(\alpha^2 A)^{\frac{1}{1-\alpha}} - \rho\right] \equiv \gamma, \tag{2.13}$$

is time independent.

i) The equilibrium production in (2.6) can also be written $Y = \bar{A}N$, where $\bar{A} \equiv AX_m^{\alpha}L^{1-\alpha}$. Thus the aggregate production is of AK-type, but with knowledge (the number of varieties) as the capital variable instead of physical capital. In addition, the real interest rate is time independent in equilibrium. It follows that we have a reduced-form AK-style model. We know that a general property of reduced-form AK models in a Ramsey framework is that the capital variable in equilibrium must grow at the same rate as consumption, the rate γ. Otherwise the households' initial consumption level would be so low that the transversality condition is not satisfied or so high that the No-Ponzi-Game condition is not satisfied.

Since both Y and X are proportional to N, they also grow at the rate γ. And since (2.8) implies $\gamma = \dot{N}/N = R/(\eta N)$, also R is proportional N and grows at the rate γ.

Two parameter restrictions are needed to ensure that this equilibrium path with positive R&D can exist. The parameters in (2.13) must be such that, first, γ is indeed positive, second, the inequality

$$\rho > (1-\theta)\gamma$$

holds, ensuring that households' discounted utility is bounded. Under these conditions the model generates fully endogenous growth.

j) Employment in the manufacturing-goods sector is now $L' = (1-s)L$. With this L' replacing L in the above formulas everything goes through. Thus, along an equilibrium path

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r'-\rho) = \frac{1}{\theta}\left[(\frac{1}{\alpha}-1)\frac{(1-s)L}{\eta}(\alpha^2 A)^{\frac{1}{1-\alpha}} - \rho\right] \equiv \gamma', \tag{2.14}$$

and this is also the growth rate of N_t. The tax on consumption at a constant rate $\tau > 0$ is non-distortionary since there is no utility from leisure in the model.

k) A higher s implies lower employment in the growth-generating sector. A lower employment in this sector implies higher cost per invention per employed in the sector. This weakens the incentive to do R&D. In this model the result is a lower growth rate forever. This reflects the controversial "strong" scale effect (scale effect on growth), typical for innovation-based growth models with fully endogenous growth: a larger economy, as measured by the size of the labor force, implies a higher growth rate. The tendency to scale effects, either on growth or just levels, in innovation-based growth models derives from knowledge being a *non-rival* good.

ℓ) A likely effect of the civil servants providing rule-of-law and social-trust services is that a higher s leads to higher A. The parameter A need not have a narrow technological interpretation, but may reflect "quality of institutions" in the economy. A likely effect of the civil servants providing technical-scientific services is that a higher s leads to lower

η, which measures private research costs. As (2.14) indicates, these two positive growth effects of a higher s partially or fully offset the negative effect through a lower L'.

m) True. There is scope for Pareto improvement in the economy because of the monopoly pricing of intermediate goods. This pricing above the marginal cost of supplying these productive goods results in inefficiently low use of these goods. Since the incentive to invest in R&D depends on expected future profits, which in turn depend on the size of the markets for intermediate goods, the monopoly pricing implies not only a static distortion but also a dynamic distortion (too little R&D in the economy).

3. Solution to Problem 3 (10 %)

a) False. Arrow's learning-by-investing model, which is based on a general neoclassical aggregate production function, predicts convergence to a steady state where the share of capital income in national income is constant. To obtain this long-run result the aggregate production function need not be Cobb-Douglas.

b) In the model type of Problem 2, called the "lab-equipment" model type, a part, R_t, of aggregate output is simply invested in $R\&D$. This reflects an assumption that the R&D technology is essentially similar to the manufacturing technology. Moreover, there is no intertemporal knowledge spillover in this model. Because of the strong scale effect, the model assumes a non-growing population.

Paul Romer (1990) and others assume the manufacturing and R&D technologies are not the same. In Romer's model the number of new varieties invented per time unit is

$$\begin{aligned} \dot{N} &= \tilde{\mu} L_N, \qquad \tilde{\mu} = \mu N, \qquad \mu > 0, \\ L_Y + L_N &= L = \text{ labor force (constant).} \end{aligned} \tag{3.1}$$

At the economy-wide level there is an intertemporal knowledge spillover, a positive externality which along with monopoly pricing results in too little R&D. Because Romer's growth engine has constant returns wrt. the producible input, N, his model generates fully endogenous growth and features a strong scale effect.

Charles Jones (1995):

$$\begin{aligned} \dot{N} &= \tilde{\mu} L_N, \qquad \tilde{\mu} = \mu N^{\varphi} L_N^{\lambda-1}, \qquad \mu > 0, 0 < \lambda \leq 1, \varphi < 1, \\ L_Y + L_N &= L. \end{aligned}$$

Here the growth engine has decreasing returns wrt. N. Hence "only" semi-endogenous growth is generated. Moreover, the parameter λ allows for a likely congestion effect in research (duplication of effort). There is no strong scale effect, but a weak scale effect (a level effect) reflecting the non-rival character of knowledge.

In Aghion and Howitt's quality ladder model (1962, 1968) successful outcomes of R&D arrive randomly with a Poisson arrival rate proportional to the R&D input. This presence of uncertainly in R&D is an attractive aspect ignored by the other approaches.

—